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Extensions of the Dirichlet–Jordan Criterion to a General Class of Orthogonal Polynomial Expansions

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1. INTRODUCTION

The well-known Dirichlet–Jordan convergence criterion for Fourier series states that the trigonometric Fourier series of a 2π-periodic function f having bounded variation converges to $\frac{1}{2}\{f(x+0)+f(x-0)\}$ for every x and this convergence is uniform on every closed interval on which f is continuous [14, Theorem 2.8.1].

A generalization of this criterion was given by Hardy and Littlewood who proved the following:

THEOREM 1.1 [11]. *Let $1 \leqslant p < \infty$. Suppose f is a 2π-periodic function in $L^p[-\pi, \pi]$ and*

$$\int_{-\pi}^{\pi} |f(x+h)-f(x)|^p \, dx = O(|h|), \qquad h > 0. \tag{1.1}$$

Then the trigonometric Fourier series of f is convergent to f at every Lebesgue point of f.

To demonstrate the fact that this is a generalization of the Dirichlet–Jordan convergence criterion, Hardy and Littlewood gave the following characterization of the functions of bounded variation:

THEOREM 1.2 [10]. *A 2π-periodic function f is equivalent to a function of bounded variation if and only if f satisfies* (1.1) *with* $p = 1$.

Analogues of the Dirichlet–Jordan convergence criterion are also known for series in polynomials orthogonal with respect to certain weights with support in $[-1, 1]$ ([2, 3]). The main difficulty here is the lack of an explicit closed expression for the Dirichlet kernel as in the case of trigonometric

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Fourier series. When the support of the weight function is unbounded, it is sometimes necessary also to find the "right" substitute for functions of bounded variation.

In 1974, Freud [4] proved an analogue of the Dirichlet–Jordan criterion for series in polynomials orthogonal with respect to weight functions supported on the whole real line. He considers weights of the form $w_Q(x) = \exp(-Q(x))$, where Q satisfies certain technical conditions. We denote by $\{p_k\}$ the system of polynomials orthonormal with respect to w_Q^2. The orthonormal Fourier series of f is then defined by

$$\Sigma a_k(w_Q^2, f)\, p_k(w_Q^2, x), \tag{1.2}$$

where

$$a_k(w_Q^2, f) := \int_{-\infty}^{\infty} f(t)\, p_k(w_Q^2, t)\, w_Q^2(t)\, dt. \tag{1.3}$$

Let the partial sums of (1.2) be denoted by s_n:

$$s_n(f, x) := s_n(w_Q^2, f; x) := \sum_{k=0}^{n-1} a_k(w_Q^2, f)\, p_k(w_Q^2, x). \tag{1.4}$$

Freud's result can now be described as follows:

THEOREM 1.3. *Suppose f is continuous on R, is of bounded variation on every compact subinterval of R and satisfy*

$$\int_{-\infty}^{\infty} w_Q\, |df| < \infty. \tag{1.5}$$

Then

$$\lim_{n\to\infty} w_Q(x)\, |s_n(w_Q^2, f, x) - f(x)| = 0 \qquad \text{uniformly on } R. \tag{1.6}$$

In particular, $s_n(w_Q^2, f, x)$ converges uniformly on compact subsets of R.

The main objective of this paper is to give a characterization of the conditions of Theorem 1.3 in terms of a modified modulus of continuity which was found suitable in the theory of weighted polynomial approximation. [5, 6, 8]. This characterization will be analogous to Theorem 1.2. We shall also demonstrate how Freud's proof of Theorem 1.3 can be modified to give a generalization of Theorem 1.3 similar to Theorem 1.1. Hardy and Littlewood used deep function theoretic techniques in the proof of their theorems. We use instead the theory of weighted polynomial approximation.

The main results are stated in the next section and the proofs are given in Section 3.

2. Main Results

We consider weight functions of the form $w_Q(x) := \exp(-Q(x))$, where Q satisfies the following conditions:

(1) Q is an even, convex function on R, continuously differentiable on $(0, \infty)$.

(2) $Q'(x) \to \infty$ as $x \to \infty$.

(3) There exists a constant A_0 such that for every δ, $0 < \delta < A_0$, one can find the least positive number x_δ such that

$$Q'(x_\delta) = \delta^{-1}. \tag{2.1}$$

We assume that

$$\delta Q'(x_\delta + \delta) \leqslant C_1 \tag{2.2}$$

for some constant C_1.

Examples of such weight functions include $\exp(-|x|^\alpha)$, $\alpha > 1$. In [7], Freud introduced an expression for the first order modulus of continuity suitable for the study of polynomial approximation with such weight functions. Let $1 \leqslant p \leqslant \infty$, $w_Q f \in L^p(R)$, $0 < \delta < A_0$ and set

$$\omega(L^p, Q, f, \delta) := \sup_{|t| \leqslant \delta} \| w_Q(x+t) f(x+t) - w_Q(x) f(x) \|_p + \delta \| Q_\delta'(x)\, w_Q(x) f(x) \|_p, \tag{2.3}$$

where

$$Q_\delta'(x) := \min\{\delta^{-1}, |Q'(x)|\}. \tag{2.4}$$

Then the first order modulus of continuity of f is defined by

$$\Omega(L^p, Q, f, \delta) := \inf_{a \in R} \omega(L^p, Q, f - a, \delta). \tag{2.5}$$

This modulus of continuity enables us to characterize functions of bounded variation in the sense of Theorem 1.3.

THEOREM 2.1. (a) *If f is a function having bounded variation on compact intervals and if*

$$\int w_Q\,|df| < \infty, \tag{2.6}$$

then, for every δ, $0 < \delta < A_0$,

$$\Omega(L^1, Q, f, \delta) = O(\delta). \tag{2.7}$$

(b) *Conversely, if* (2.7) *holds then f is almost everywhere equal to a function having bounded variation on compact intervals and* (2.6) *holds.*

To state our results concerning the convergence of orthogonal polynomial series, we need to impose more restrictive conditions on the weight function. Individual results will actually be true under slightly weaker conditions, but the following conditions certainly suffice.

(A) Q is even, convex and is in $C^2(0, \infty)$.

(B) Q'' is increasing on $(0, \infty)$.

(C) $$1 \leqslant c_2 \leqslant \frac{xQ''(x)}{Q'(x)} \leqslant c_3 \tag{2.8}$$

for some constants c_2 and c_3.

All of these conditions are satisfied by $w_Q(x) = \exp(-|x|^\alpha)$ when $\alpha \geqslant 2$.

We can now state the analogue of the Hardy–Littlewood criterion (Theorem 1.1).

THEOREM 2.2. *Let $1 \leqslant p < 2$, f be continuous on R, $w_Q f \in L^p(R)$ and*

$$\Omega(L^p, Q, f, \delta) = O(\delta^{1/p}). \tag{2.9}$$

Then

$$\|w_Q(f - s_n(f))\|_\infty \to 0 \qquad as \quad n \to \infty \tag{2.10}$$

where $s_n(f)$ is defined in (1.4).

If $p \geqslant 2$, and

$$\Omega(L^p, Q, f, \delta) = o(\delta^{(p-1)/p}) \tag{2.11}$$

then (2.10) *holds.*

We believe that if $p \geqslant 2$, (2.9) would not, in general, imply (2.10).

3. Proofs

We adopt the following notation. $A \ll B$ means that there is a positive constant c independent of all obvious variables such that $A \leqslant cB$. $A \sim B$ means that $A \ll B$ and $B \ll A$.

Proof of Theorem 2.1(a). Let $g(x) := f(x) - f(0)$, $h > 0$. Then

$$\begin{aligned}
&\int_0^\infty |w_Q(x+h)\, g(x+h) - w_Q(x)\, g(x)|\, dx \\
&\quad = \int_0^\infty \left| \int_x^{x+h} [-Q'(t)\, w_Q(t)\, g(t)\, dt + w_Q(t)\, df(t)] \right| dx \\
&\quad \leqslant \int_0^\infty \left[\int_x^{x+h} Q'(t)\, w_Q(t)\, |g(t)|\, dt + \int_x^{x+h} w_Q(t)\, |df(t)| \right] dx \\
&\quad = \int_0^\infty Q'(t)\, w_Q(t)\, |g(t)| \int_{\max(0,t-h)}^t dx\, dt \\
&\qquad + \int_0^\infty w_Q(t) \int_{\max(0,t-h)}^t dx\, |df(t)| \\
&\quad \leqslant hT + h\int_0^\infty w_Q(t)\, |df(t)|, \qquad (3.1)
\end{aligned}$$

where

$$\begin{aligned}
T &:= \int_0^\infty Q'(t)\, w_Q(t)\, |g(t)|\, dt \\
&= \int_0^\infty Q'(t)\, w_Q(t) \left| \int_0^t df(u) \right| dt \\
&\leqslant \int_0^\infty Q'(t)\, w_Q(t) \int_0^t |df(u)|\, dt \\
&= \int_0^\infty \left(\int_u^\infty Q'(t)\, w_Q(t)\, dt \right) |df(u)| \\
&= \int_0^\infty w_Q(u)\, |df(u)|. \qquad (3.2)
\end{aligned}$$

Thus

$$\int_0^\infty |w_Q(x+h)\, g(x+h) - w_Q(x)\, g(x)|\, dx \leqslant 2h \int_0^\infty w_Q(u)\, |df(u)|. \qquad (3.3)$$

A similar estimate can be obtained also when the integrals extend on $(-\infty, 0]$. Then, if $h > 0$,

$$\int_{-\infty}^{\infty} |w_Q(x+h)\, g(x+h) - w_Q(x)\, g(x)|\, dx \leqslant 2\,|h| \int_{-\infty}^{\infty} w_Q(u)\,|df(u)|. \tag{3.4}$$

Translation invariance of the integral then gives (3.4) for $h < 0$ as well. Similarly, from (3.2), we get

$$\|Q'_\delta w_Q\, g\|_1 \leqslant \|Q' w_Q\, g\|_1 \leqslant \int_{-\infty}^{\infty} w_Q(t)\,|df(t)|. \tag{3.5}$$

From (3.4), (3.5), (2.3), (2.5) we have

$$\Omega(L^1, Q, f, \delta) \leqslant \omega(L^1, Q, g, \delta) \leqslant 3\delta \int_{-\infty}^{\infty} w_Q(t)\,|df(t)|. \quad \blacksquare \tag{3.6}$$

To prove Theorem 2.1(b), we need some preliminary results. For $\delta < A_0$, define x_δ by (2.1). Set

$$\phi_\delta(x) := \begin{cases} w_Q(x) f(x) & \text{if } \ |x| \leqslant x_\delta \\ 0 & \text{otherwise} \end{cases} \tag{3.7}$$

and

$$\psi_\delta(x) := \delta^{-1} w_Q^{-1}(x) \int_0^\delta \phi_\delta(x+t)\, dt. \tag{3.8}$$

Then, clearly, there is a set E such that $R \backslash E$ is a set of measure zero and

$$\psi_\delta(x) \to f(x) \qquad \text{as} \quad \delta \to 0 \text{ if } x \in E. \tag{3.9}$$

Moreover, we have the following estimates [7]:

$$\|w_Q \psi'_\delta\|_1 \ll \delta^{-1} \Omega(L^1, Q, f, \delta) \leqslant M, \tag{3.10}$$

$$\|w_Q(\psi_\delta - f)\|_1 \ll \Omega(L^1, Q, f, \delta) \leqslant \delta M, \tag{3.11}$$

where

$$M := \sup_{0 < \delta < A_0} \delta^{-1} \Omega(L^1, Q, f, \delta). \tag{3.12}$$

LEMMA 3.1. *Let* $0 \leqslant x_1 < \cdots < x_n < \infty$. $x_0, x_1, \ldots, x_n \in E$. *Then*

$$\sum_{k=1}^{n} w_Q(x_k)\,|f(x_k) - f(x_{k-1})| \ll M. \tag{3.13}$$

Proof. Choose $\delta > 0$ so small that

$$|\psi_\delta(x_i) - f(x_i)| \leqslant M \left(\sum_{k=1}^{n} w_Q(x_k) \right)^{-1}, \qquad i = 0, 1, ..., n. \tag{3.14}$$

This is possible in view of (3.9). By replacing w_Q by an equivalent weight if necessary (cf. [9]), we may assume without loss of generality that w_Q is decreasing on $(0, \infty)$. Then

$$\begin{aligned} &\sum_{k=1}^{n} w_Q(x_k)\, |f(x_k) - f(x_{k-1})| \\ &\quad \leqslant M + \sum_{k=1}^{n} w_Q(x_k) [|f(x_k) - \psi_\delta(x_k)| + |\psi_\delta(x_k) - \psi_\delta(x_{k-1})| \\ &\quad + |f(x_{k-1}) - \psi_\delta(x_{k-1})|]. \end{aligned} \tag{3.15}$$

Using (3.14), (3.10) and the fact that w_Q is decreasing, we get from (3.15) that

$$\begin{aligned} &\sum_{k=1}^{n} w_Q(x_k)\, |f(x_k) - f(x_{k-1})| \\ &\quad \leqslant M + \sum_{k=1}^{n} w_Q(x_k) \int_{x_{k-1}}^{x_k} |\psi'_\delta(t)|\, dt \\ &\quad \leqslant M + \sum_{k=1}^{n} \int_{x_k}^{x_k} w_Q(t)\, |\psi'_\delta(t)|\, dt \leqslant M. \quad \blacksquare \end{aligned} \tag{3.16}$$

Proof of Theorem 2.1(b). In view of Lemma 3.1, f is of bounded variation on $E \cap [0, R]$ for every $R > 0$. It follows by a standard argument (cf. [1, p. 73]) that f is almost everywhere equal to a function having bounded variation on compact subintervals of $[0, \infty)$. Similar argument gives the same result for $(-\infty, 0]$. Hence f is almost everywhere equal to a function having bounded variation on compact subintervals of R. We shall identify f with this function. Let

$$V(x) := \int_0^x |df(t)|. \tag{3.17}$$

Suppose $0 \leqslant x_0 \leqslant \cdots \leqslant x_n$ are arbitrary and $\varepsilon > 0$ is given. Find $x_k \leqslant x_{k,0} \leqslant \cdots \leqslant x_{k+1}$ $(k = 0, 1, ..., n-1)$ such that for each $k = 0, 1, ..., n-1$

$$\sum_{r=1}^{m_k} |f(x_{k,r}) - f(x_{k,r-1})| \geqslant V(x_{k+1}) - V(x_k) - \varepsilon. \tag{3.18}$$

Then, using Lemma 3.1,

$$\begin{aligned}
&\sum_{k=1}^{n} w_Q(x_k)[V(x_k)-V(x_{k-1})] \\
&\quad\leqslant \varepsilon \sum_{k=1}^{n} w_Q(x_k) + \sum_{k=1}^{n} \sum_{r=1}^{m_{k-1}} w_Q(x_k)\,|f(x_{k-1,r})-f(x_{k-1,r-1})| \\
&\quad\leqslant \varepsilon \sum_{k=1}^{n} w_Q(x_k) + \sum_{k=1}^{n} \sum_{r=1}^{m_{k-1}} w_Q(x_{k-1,r})\,|f(x_{k-1,r})-f(x_{k-1,r-1})| \\
&\quad\ll \varepsilon \sum_{k=1}^{n} w_Q(x_k) + M. \qquad (3.19)
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary,

$$\sum_{k=1}^{n} w_Q(x_k)[V(x_k)-V(x_{k-1})] \ll M. \qquad (3.20)$$

Using the continuity of w_Q, it is now elementary to see that

$$\int_0^\infty w_Q\,|df| \ll M. \qquad (3.21)$$

Similarly, the integral over $(-\infty, 0]$ can be estimated, thus completing the proof. ∎

In the remaining part of the paper we assume the stronger conditions on the weight function stated in Section 2. For $w_Q f \in L^p(R)$ and integer $n \geqslant 1$, set

$$v_n(f, x) := v_n(w_Q^2, f, x) := \frac{1}{n} \sum_{m=n+1}^{2n} s_m(w_Q^2, f, x), \qquad (3.22)$$

$$\varepsilon_n(p, w_Q, f) = \varepsilon_n(p, f) := \inf \| w_Q[f-P]\|_p, \qquad (3.23)$$

where the inf is taken over all polynomials P of degree at most n. The class of all such polynomials will be denoted by π_n. We need to recall a few facts which we summarize in the following lemma.

LEMMA 3.2. *Let* $1 \leqslant p,\ r \leqslant \infty,\ 0 < \alpha \leqslant 1,\ w_Q f \in L^p(R)$ *and* n *(an integer) be* $\geqslant 1$. *Then*

(a) [5]

$$\|f - v_n(f)\|_p \ll \varepsilon_n(p, f); \qquad (3.24)$$

(b) [13]

$$\|f - s_n(f)\|_2 = \varepsilon_n(2, f); \tag{3.25}$$

(c) [12] *if* $P \in \pi_n$, *then*

$$\|w_Q P\|_r \ll \left(\frac{n}{q_n}\right)^{|1/p - 1/r|} \|w_Q P\|_p, \tag{3.26}$$

where q_n *is the least positive solution of the equation* $xQ'(x) = n$:

$$q_n Q'(q_n) = n, \tag{3.27}$$

(d) [5]

$$\varepsilon_n(p, f) \ll \Omega\left(L^p, Q, f, \frac{q_n}{n}\right); \tag{3.28}$$

(e) [12] *if*

$$\Omega(L^p, Q, f, \delta) = O(\delta^\alpha) \tag{3.29}$$

and

$$\beta := \alpha - \left|\frac{1}{p} - \frac{1}{r}\right| > 0,$$

then

$$\Omega(L^r, Q, f, \delta) = O(\delta^\beta); \tag{3.30}$$

(f) *part* (e) *is true if* 0 *in* (3.29) *and* (3.30) *is replaced by o.*

The proof of part (f) is almost identical to that of part (e). The proof of Theorem 2.2 is now simple; in fact, it is the same as Freud's original proof of Theorem 1.3.

Proof of Theorem 2.2. In view of (3.24) we need only to estimate $v_n(f) - s_n(f)$. Since this is in π_{2n}, we get using (3.26), (3.24), (3.25),

$$\begin{aligned} &\|w_Q(v_n(f) - s_n(f))\|_\infty \\ &\quad \ll \sqrt{\frac{n}{q_n}}\, \|w_Q(v_n(f) - s_n(f))\|_2 \\ &\quad \leqslant \sqrt{\frac{n}{q_n}}\, [\|f - v_n(f)\, w_Q\|_2 + \|w_Q(f - s_n(f))\|_2] \\ &\quad \ll \sqrt{\frac{n}{q_n}}\, \varepsilon_n(2, f). \end{aligned} \tag{3.31}$$

Let $m := [n/2]$. Then $v_m(f) \in \pi_n$ and if $1 \leqslant p < 2$,

$$\begin{aligned}\| w_Q(v_n(f) - s_n(f))\|_\infty &\ll \sqrt{\frac{n}{q_n}} \| w_Q(f - v_m(f))\|_2 \\ &\leqslant \sqrt{\frac{n}{q_n}} \| w_Q(f - v_m(f))\|_p^{p/2} \| w_Q(f - v_m(f))\|_\infty^{1-p/2} \\ &\ll \sqrt{\frac{n}{q_n}} [\varepsilon_m(p, f)]^{p/2} [\varepsilon_m(\infty, f)]^{1-p/2}. \qquad (3.32)\end{aligned}$$

In view of Lemma 3.2(e), (d) and the fact that [5]

$$q_{2m} \sim q_m \qquad (3.33)$$

we get

$$\| w_Q(v_n(f) - s_n(f))\|_\infty \ll [\varepsilon_m(\infty, f)]^{1-p/2}. \qquad (3.34)$$

Hence, if $w_Q f \in C_0(R)$, (3.34) and Lemma 3.2(a) imply (2.10).

In the case when $p \geqslant 2$, (2.11), Lemma 3.2(f), (d) and (3.31) give

$$\| w_Q(v_n(f) - s_n(f))\|_\infty \to 0 \qquad \text{as} \quad n \to \infty. \qquad (3.35)$$

Thus, (2.10) holds as before. ∎

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